On idempotents and clean elements in certain ring extensions

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Noncommutative Rings and their Applications

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Definition An element of a ring is called *clean* if it can be expressed as a sum of an idempotent and a unit in the ring. A ring is called *clean* if each of its elements is clean.

- **Clean Group Algebras**

Observation Suppose KH is clean for every finitely generated subgroup H of a group G. Then KG is clean.

Observation If G is a locally finite group, then KG is clean.

- – **Notation**

  $\Delta(G) = \{x \in G \mid [G : C_G(x)] < \infty\}$

  $\Delta^+(G) = \{x \in \Delta(G) \mid o(x) < \infty\}$

  $= \{x \in G \mid [G : C_G(x)] < \infty$ and $o(x) < \infty\}$

  $\Lambda(G) = \{x \in G \mid [G : C_G(x)] = \text{l.f.}\}$

  (Recall $[G : H] = \text{l.f.}$ means $[K : K \cap H] < \infty$ for all finitely generated subgroups K of G.)

  $\Lambda^+(G) = \{x \in \Lambda(G) \mid o(x) < \infty\}$

  For any group G and any element $\sum_{g \in G} \alpha_g g$ in KG,

  $$\left(\sum_{g \in G} \alpha_g g\right)^* = \sum_{g \in G} \alpha_g g^{-1}$$

  D_∞ : Infinite Dihedral group, that is, the group generated by two elements a and b where a is of infinite order and $b^2 = 1$, $ab = ba^{-1}$.

  For any ring R,

  $U(R)$: Group of all units in R.

Definition A group G is called *polycyclic* if G has a finite subnormal series

$$(1) = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G \qquad (1)$$

such that each quotient G_{i+1}/G_i is cyclic. If G_{i+1}/G_i is either cyclic or finite then G is called *polycyclic-by-finite*.

Definition A group G is called *nilpotent* if G has a central series, that is, a normal series

$$(1) = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

such that each quotient G_{i+1}/G_i is contained in the center of G/G_i for all i.

Proposition Let K be a field and G be a polycyclic-by-finite group. Then KG is clean if and only if G is finite. In particular, for any field K, the group algebra KD_∞ of the infinite dihedral group, D_∞, is not clean.

Proposition Let G be a finitely generated solvable group such that the group algebra KG is clean. Then G is finite.

Proof Proof by induction on the solvability length of G.

Corollary Let G be a group with a torsion FC subgroup. Suppose H is an FC normal subgroup of G such that $\frac{G}{H}$ is finitely generated solvable. If KG is clean then G is locally finite.

Theorem Let G be a nilpotent or FC or locally FC group. Then KG is clean if and only if G is locally finite.

Proposition Let G be a residually finite p-group and K be a field of characteristic $p > 0$. Then KG is clean if and only if it is local and hence G is torsion. (Recall a group G is called residually finite if for every $g \neq 1$ in G there exists a normal subgroup N of G such that $g \notin N$ and $\frac{G}{N}$ is finite.)

Lemma Let G be a residually finite p-group and K be a field of characteristic $p > 0$. Then KG has no non trivial idempotents.

Example Let G be an infinite cyclic group. Since G is a free group, it is residually a finite p-group for all primes p. However, KG is not clean. Note that, in this example, for every non trivial subgroup H of G, $K[\frac{G}{H}]$ is clean but KH is not clean.

Proposition Let KG be a prime group algebra in which all idempotents are central. If KG is clean then KG is local.

Corollary Let KG be a prime group algebra such that supporting group of all idempotents in KG is finite. If KG is clean then G is torsion.

- **Clean elements in KD_∞**
  - D_∞ : Infinite Dihedral group, that is, the group generated by two elements a and b where a is of infinite order and $b^2 = 1$, $ab = ba^{-1}$.
  - A : $\langle a \rangle$, infinite cyclic group generated by a.
  - For any group G and any element $\sum_{g \in G} \alpha_g g$ in KG,

$$\left(\sum_{g \in G} \alpha_g g \right)^* = \sum_{g \in G} \alpha_g g^{-1}$$

- **Clean elements in KD_∞**
  - Some remarks
    - * For any unit $\alpha+\beta b$ in KD_∞, $\alpha\alpha^*-\beta\beta^* \in K\setminus\{0\}$.
    - * For any idempotent $\alpha+\beta b$ in KD_∞, $\alpha+\alpha^*=1$ and $\alpha\alpha^*=\beta\beta^*$.
    - * Any idempotent e in KD_∞ has the form $2^{-1}+\alpha_1+\beta b$ where $\alpha_1=-\alpha_1^*$ and $\alpha_1\alpha_1^*-\beta\beta^*$ is a nonzero element of K.
    - * If $char(K)=2$, then KD_∞ has no nontrivial idempotents and hence any element α in KD_∞ ($char(K)=2$) is clean if and only if either it is a unit or $\alpha-1$ is a unit.

- **Clean elements in KD_∞**

Theorem Let K be a field of characteristic not equal to 2 and let $\alpha \in KA \subset KD_\infty$.

1. If $\alpha = a + \beta$, where $\beta = -\beta^*$ and $a \in K$ then α is clean in KD_∞ if and only if $a \neq 0, 1$.

2. If $\alpha = \alpha^*$, then α is clean in KD_∞ if and only if $\alpha \in K$.

Remark. Same argument can be used if K is replaced with a commutative domain in which 2 is invertible.

Remark. It can be similarly proved that

1. If $0 \neq \alpha = \alpha^* \in KA$ then $\alpha(1+b)$ is clean if and only if $\alpha \in K$.

2. If $\alpha^* = -\alpha \in KA$ then $\alpha(1+b)$ is never clean.

- **Clean elements in Polynomial Rings**

Known $R[x]$ is not clean. (Infact, $x \in R[x]$ is not clean.)

Observation If R is reduced ring then clean elements in $R[x]$ are in R.

Proposition $Cl(R[X]) = Cl(R)$ if and only if R is reduced where $Cl(R)$ denotes the set of clean elements in R .

Proof If $a \in R$ is a nilpotent element, then $u = 1 + ax$ is invertible in $R[x]$, so $Cl(R[X]) \neq Cl(R)$ in this case. If R is reduced then $U(R[x]) = U(R)$ and $E(R[x]) = E(R)$. Thus also $Cl(R[x]) = Cl(R)$.

- **Idemptents in Polynomial Rings and other ring extensions**

For a unital ring R,

$E(R)$: Set of all idempotents in R.

$J(R)$: Jacobson radical of R.

$B(R)$: Prime radical of R.

Lemma Let R be a ring and $e(x) = \sum_{i=0}^{\infty} e_i x^i \in R[[x]]$ be an idempotent. If $e_0 e_i = e_i e_0$, for every $i \geq 1$, then $e(x) = e_0$. In particular, if R is abelian, then $E(R[[x]]) = E(R[x]) = E(R)$.

Proposition Let S_n denote one of the following rings $R[x_1, \ldots, x_n]$, $R[[x_1, \ldots, x_n]]$ and $R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. If e is a central idempotent of S_n, then $e \in R$.

Corollary (Bass) Let K be a commutative ring and G an abelian group with the torsion part H. Then any idempotent of KG belongs to KH.

Theorem For a ring R, the following conditions are equivalent

1. R is abelian.

2. Idempotents of R commute with units of R.

3. $E(R[[x]]) = E(R)$.

4. $E(R[x, x^{-1}]) = E(R)$.

5. $E(R[x]) = E(R)$.

6. There exists $n \geq 1$ such that $R[x]$ does not contain idempotents which are polynomials of degree n.

Remark Each of the statements in the above proposition is equivalent to the statement

The rings $R[x]$, $R[[x]]$, $R[x, x^{-1}]$ are all abelian.

Corollary Let S denote one of the rings $R[x_1, \ldots, x_n]$, $R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$, $R[[x_1, \ldots, x_n]]$. Then R is is abelian if and only if S is abelian if and only if $E(S) = E(R)$.

Proposition Let M be an additive monoid with neutral element 0 and suppose $R = \bigoplus_{m \in M} R_m$ is an M-graded ring. Then

1. If R_0 is abelian and $E(R) = E(R_0)$, then R is an abelian ring.

2. Suppose $M = \mathbb{Z}$. Then R is abelian if and only if R_0 is abelian and $E(R) = E(R_0)$.

Remark The analogue of the statement 'the polynomial ring $R[x]$ is abelian if and only if R is abelian' does not hold for $\mathbb{Z}$-graded rings. Indeed, if K is a field and $e = (1, 0) \in K \times K = R_0$ then R_0 is commutative but the idempotent e is not central in $R = R_0[x; \sigma]$, where σ is the automorphism of $R_0 = K \times K$ switching components.

Example Let $R = M_2(\mathbb{Z}_4)$. $E = \begin{pmatrix} 1+x^2 & x+x^3 \\ 3x & 3x^2 \end{pmatrix}$.

E is an idempotent in $R[x]$.

E is conjuage to the idempotent $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in R. Indeed, $E = P^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P$ where $P = \begin{pmatrix} 1 & x \\ x & 1+x^2 \end{pmatrix}$.

Remark Note that if $e(x) = \sum_{i=0}^{n} e_i x^i \in R[x]$ is an idempotent then $e(x) = e_0 + b$ where e_0 is an idempotent in R.

Proposition Let e, b, u be elements in a ring R such that $e^2 = e$ and $u = 2e - 1$. Then the following conditions are equivalent.

1. $e + b$ is idempotent.
2. $be + eb + b^2 = b$.
3. $(1 + bu)e = (e + b)(1 + bu)$.

   Moreover, if one of the equivalent statements holds then

4. $bu + ub = -2b^2$.
5. $b^2u = ub^2$ and $(1 + bu)(1 + ub) = (1 + ub)(1 + bu) = 1 - b^2$.
6. $1 + bu$ is invertible iff $1 + ub$ is invertible iff $1 - b^2$ is invertible.

7. $(1 + 2ub)(1 + 2bu) = 1$ and $b^2u = ub^2$.

Corollary Let $e, b \in R$ be such that $e, e+b \in E(R)$. If $1-b^2$ is invertible, then e and $e+b$ are conjugate. In particular this holds when either b is nilpotent or $b \in J(R)$ - the Jacobson radical of R.

Remark It is possible for two idempotents to be conjugates without $1-b^2$ being invertible.

Example Let $R = M_2(\mathbb{Z}_4)$. $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} x^2 & x+x^3 \\ 3x & 3x^2 \end{pmatrix}$. Then $e+b = \begin{pmatrix} 1+x^2 & x+x^3 \\ 3x & 3x^2 \end{pmatrix}$ which is a conjugate of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Here $1-b^2 = diag(1-3x^2, 1-3x^2)$ which is not invertible.

Remark Let $e, b \in R$ be such that $e, e' = e+b \in E(R)$. Let $u = 2e - 1$.

1. If $k \in \mathbb{N}$ is odd, then $(eb^k + eb^{k+1})e = 0$.

2. If $k \in \mathbb{N}$ is odd, then $e + eb + eb^2 + \cdots + eb^{k-1}$ is an idempotent.

3. If $b^k = 0$, then $e + eb + eb^2 + \cdots + eb^{k-1}$ is always an idempotent.

4. $e - (1 + 2ub)b$ is an idempotent and we have $(1 + ub)e = (e - (1 + 2ub)b)(1 + ub)$.

5. $e + 2b(1 + ub)$ is an idempotent and we have $(e + b)(1 + ub) = (1 + ub)(e + 2b(1 + ub))$.

6. If $be = eb$, then $b = b^3$. In particular, b^2 is an idempotent.

Theorem Any idempotent f of $R[[x]]$ is conjugate to its constant term. Thus, in particular, any idempotent of $R[[x_1, \ldots, x_n]]$ is conjugate to an idempotent of R.

Corollary Let R be a ring. Then

1. Any idempotent of $R[x]/(x^n)$ is conjugated to an idempotent of R.

2. Any idempotent of the upper triangular matrix ring $A_n(R)$ of $n \times n$ matrices over R is conjugated to a diagonal idempotent matrix.

3. If S is another ring and ${}_R M_S$ is an (R, S)-bimodule, then any idempotent of the ring $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is conjugate to an idempotent of $R \oplus S$.

Corollary Let R be any ring and $e(x) = e + cx^n \in R[x]$ be an idempotent, where $e, c \in R$ and $n \geq 1$. Then $e(x)$ is conjugate in $R[x]$ to $e = e^2 \in R$. In particular, every idempotent of $R[x]$ having degree one is conjugate to an idempotent of R.

Proof Note that $e(x)$ is an idempotent implies e is an idempotent and $b = cx^n$ a nilpotent element.

Question What can we say about polynomials of the type $e + bx^m + cx^n \in R[x]$ $(m \neq n)$?

Answer In general, No.

Definition A ring R is called 2-primal if $\dfrac{R}{B(R)}$ is reduced, equivalently, the set of all nilpotent elements of R coincides with the prime radical $B(R)$ of R.

Theorem Suppose R is a 2-primal ring. Then any idempotent of $R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ as well as of $R[x_1, \ldots, x_n]$. is conjugated to an idempotent of R.

Definition An idempotent e of a ring R is called right semicentral if $er = ere$, for all $r \in R$. Left semicentral idempotents are defined similarly.

Proposition Let $T \subset S$ be a ring extension and $e, f \in T$ be right semicentral idempotents of S. If e, f are conjugate in S, then:

1. $e = ef$ and $f = fe$;

2. e and f are conjugate in T.

Theorem Let f be a right (resp. left) semicentral idempotent of $R[x]$. Then f is conjugate to the free term of f.

Definition Two elements e, e' of a ring R are called equivalent if there exist invertible elements $p, q \in R$ such that $e' = peq$.

Corollary Let e, e' be two idempotents of a ring R. Then e and e' are equivalent if and only if they are conjugate.

Definition A ring R is called projective-free if every finitely generated projective R-module is free of unique rank.

Remark A ring is projective-free precisely when it has invariant basis number (IBN for short) and every idempotent matrix is conjugate to a matrix of the form $diag(1, \ldots, 1, 0 \ldots, 0)$.

Theorem

1. Let I denote an ideal of ring R contained in the Jacobson radical $J(R)$ of R. If R/I is projective-free then R is also projective-free;

2. Every local ring R is projective-free.

Theorem (Cohn) Let R be any projective-free ring. Then the power series ring $R[[x]]$ is again projective-free.

Remark Suppose B is a ring such that the ring $B[x]$ is projective-free. Then, looking at $M_n(B)[x]$ as $M_n(B[x])$, every idempotent of $R[x]$ is conjugate to an idempotent of $R = M_n(B)$.

Definition A ring R is called an ID ring if every idempotent matrix over R is conjugated to a diagonal matrix.

Theorem Let R be a 2-primal ring such $R[x]$ is an ID-ring. Then every idempotent $e \in M_n(R)[x]$ is conjugated to a diagonal matrix of the form $diag(e_1, \ldots, e_n) \in M_n(R)$, where e_i's denote idempotents in R.

Remark Any commutative ring R such that $R/B(R)$ is a principal ideal ring fulfills the assumptions of the above theorem.

Thank You